# Analytic structure in the coupling constant plane in perturbative QCD

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## Abstract

We investigate the analytic structure of the Borel-summed perturbative QCD amplitudes in the complex plane of the coupling constant. Using the method of inverse Mellin transform, we show that the prescription dependent Borel-Laplace integral can be cast, under some conditions, into the form of a dispersion relation in the $a$-plane. We also discuss some recent works relating resummation prescriptions, renormalons and nonperturbative effects, and show that a method proposed recently for obtaining QCD nonperturbative condensates from perturbation theory is based on special assumptions about the analytic structure in the coupling plane that are not valid in QCD.

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## I. INTRODUCTION

The QCD amplitudes have a complicated analytic structure in the complex plane of the coupling constant, $\alpha_s = g^2$. As proved in [1, 2], the infinitely many multiparticle branch points at large energies result, via renormalization group invariance, in an accumulation of essential singularities near the origin $\alpha_s = 0$. Since in massless QCD the multiparticle hadronic states are generated only by a nonperturbative confinement mechanism, these singularities can show up only beyond perturbation theory. However, the perturbative amplitudes themselves are expected to have a complicated structure of singularities, due to the fact that the perturbation series is divergent and Borel non-summable. The presence of the renormalons on the real axis of the Borel plane induces singularities of the amplitudes as functions of the coupling constant.

In a recent paper [3], arguments based on analyticity in the coupling complex plane were used to suggest the possibility of calculating genuine nonperturbative quantities, like QCD condensates, from pure perturbation theory. The analytic structure and its connection with infrared renormalons were further discussed in [4, 5]. Motivated by this recent interest in the problem, we investigate in the present work the analytic structure in the complex coupling plane of the Borel-summed amplitudes in perturbative QCD. We use the mathematical techniques applied in [6, 7, 8], which allow us to express the Borel integral as a dispersion relation in the coupling plane. In the last section we shall make a few comments on the papers [3, 4, 5].

We consider for illustration the Adler function in massless QCD

$$\mathcal{D} = -Q^2 \frac{\mathrm{d}\Pi(Q^2)}{\mathrm{d}Q^2} - 1 \,, \tag{1}$$

where $\Pi(Q^2)$ is the current-current correlation function calculated for euclidian arguments $Q^2 > 0$. It is known that the perturbation expansion of $\mathcal{D}_{PT}(a)$ in powers of the renormalized coupling $a = \alpha_s(Q^2)/\pi$ is divergent and not Borel summable (see [9] and references therein). The attempts of performing the summation by a formal Borel-Laplace integral [10] encounter the difficulty that this integral is not well-defined. We consider the Borel transform $B(u)$ defined in the standard way in terms of the perturbative coefficients $d_n$ of $\mathcal{D}$:

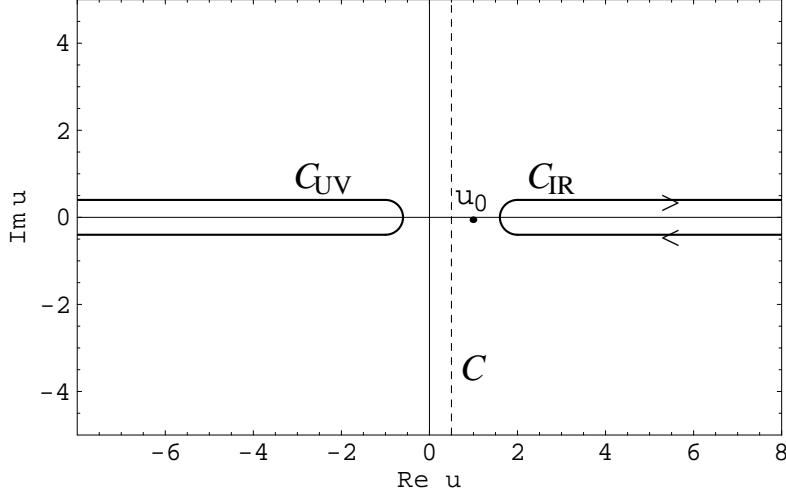$$B(u) = \sum_{n=0}^{\infty} \frac{d_n}{n!} \left( \frac{u}{\beta_0} \right)^n \,, \tag{2}$$

FIG. 1: The Borel plane for the Adler function

where $\beta_0 = (33 - 2n_f)/12$ is the first QCD beta-function coefficient with $n_f$ the number of flavors. It is known, from the $n!$ large order growth of $d_n$, that $B(u)$ has singularities (ultraviolet and infrared renormalons) on the real axis of the $u$-plane [9]. For the Adler function, the ultraviolet renormalons are placed along the range $u \leq -1$ and the infrared renormalons along $u \geq 2$ (see Fig. 1). Due to the infrared renormalons, the usual Borel-Laplace integral is not well-defined and requires an integration prescription. Defining

$$\mathcal{D}_{PT}^{(\pm)}(a) = \frac{1}{\beta_0} \int_{\mathcal{C}_\pm} e^{-\frac{u}{\beta_0 a}} B(u) \, du = \frac{1}{\beta_0} \lim_{\epsilon \to 0} \int_{0 \pm i\epsilon}^{\infty \pm i\epsilon} e^{-\frac{u}{\beta_0 a}} B(u) \, du \,, \tag{3}$$

one can adopt as prescription, for each value of $a$ with $\mathrm{Re}\, a > 0$, either $\mathcal{D}_{PT}^{(+)}(a)$ or $\mathcal{D}_{PT}^{(-)}(a)$, or a linear combination of them, with coefficients $\xi$ and $1 - \xi$ such as to correctly reproduce the known low-order expansion of $\mathcal{D}_{PT}(a)$ (which is obtained by truncating the Taylor expansion (2) at a finite order $N$). We consider in particular the principal value ($PV$) prescription

$$\mathcal{D}_{PT}^{(PV)}(a) = \frac{1}{2}[\mathcal{D}_{PT}^{(+)}(a) + \mathcal{D}_{PT}^{(-)}(a)] \,. \tag{4}$$

Once a prescription is adopted, one has a well-defined function of $a$, different prescriptions yielding different functions. In the next section we shall study the analytic properties of these functions in the complex $a$-plane.

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## II.  DISPERSION RELATIONS FOR THE BOREL SUMMED AMPLITUDE

The analytic properties of the integrals (3) with respect to the variable $a$ can be studied with standard mathematical techniques. In the present work, we use the method of inverse Mellin transform, applied for the first time in the context of Borel summation in QCD in [11, 12]. For details of the mathematical procedure used below, we refer to [6], where the same method was applied for investigating the momentum-plane analyticity structure of the Adler function in the large-$\beta_0$ limit.

The inverse Mellin transform of the function $B(u)$ is defined by [13]

$$\widehat{w}(\tau) = \frac{1}{2\pi i} \int\limits_{u_0-i\infty}^{u_0+i\infty} B(u)\, \tau^{u-1} \mathrm{d}u \tag{5}$$

and admits the inverse relation

$$B(u) = \int\limits_{0}^{\infty} \widehat{w}(\tau)\, \tau^{-u} \mathrm{d}\tau\,, \tag{6}$$

which gives $B(u)$ in the strip $-1 < \mathrm{Re}\, u < 2$ parallel to the imaginary axis [13]. The above relations are valid if the Borel transform satisfies the condition [13]

$$\frac{1}{2\pi i} \int\limits_{u_0-i\infty}^{u_0+i\infty} |B(u)|^2 \mathrm{d}u < \infty\,, \tag{7}$$

where $u_0$ is a point located on the real axis, between the branch points, $-1 < u_0 < 2$ (see Fig.1). This condition strongly restricts the asymptotic behaviour of the Borel transform, and it is not known whether it is obeyed or not in QCD. The condition (7) is however satisfied in some particular cases of physical interest. One example is the summation of a chain of diagrams in the large-$\beta_0$ (or large-$n_f$ limit), which leads to the Borel transform [14, 15]:

$$B(u) = \frac{128}{3(2-u)} \sum_{k=2}^{\infty} \frac{(-1)^k\, k}{[k^2 - (1-u)^2]^2}\,, \tag{8}$$

working in the $V$-scheme, where all the exponential factors are included in the definition of the coupling. As a second example we take the case of a finite number of renormalons with branch-point singularities, in particular the contribution to $B(u)$ of the leading infrared and ultraviolet renormalons

$$B(u) = \frac{K}{(1-u/p)^{\nu+1}} + \frac{K'}{(1+u/p')^{\nu'+1}}\,, \tag{9}$$

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where $p > 0$, $p' > 0$ and the constants $K$ and $K'$ represent the strength of the corresponding singularities. We note that for the Adler function $p = 2$, $p' = 1$, and the exponents $\nu$ and $\nu'$ were calculated in [16] and [17], respectively.

The function $\widehat{w}(\tau)$ defined in (5) can be calculated by closing the integration contour along a semi-circle at infinity in the $u$ plane and applying the theorem of residues. For $|\tau| < 1$ the contribution of the semi-circle at infinity vanishes if the contour is closed in the right half of the $u$ plane, while for $|\tau| > 1$ the contour must be closed in the left half plane. Therefore one obtains different expressions for the distribution function at $|\tau| < 1$ and $|\tau| > 1$. By Cauchy's theorem these functions, which we denote by $\widehat{w}_<(\tau)$ and $\widehat{w}_>(\tau)$, have the representations

$$\widehat{w}_<(\tau) = \frac{1}{2\pi i} \int_{\mathcal{C}_{\mathrm{IR}}} B(u)\, \tau^{u-1} \mathrm{d}u, \tag{10}$$

$$\widehat{w}_>(\tau) = \frac{1}{2\pi i} \int_{\mathcal{C}_{\mathrm{UV}}} B(u)\, \tau^{u-1} \mathrm{d}u,$$

the integration (Hankel) contours $\mathcal{C}_{\mathrm{IR}}$ and $\mathcal{C}_{\mathrm{UV}}$ being indicated in Fig. 1.

The inverse Mellin transform $\widehat{w}$ was calculated in Ref. [11] for the Adler function in the large-$\beta_0$ limit, when the Borel transform has the expression (8). In this case

$$\widehat{w}_<(\tau) = \frac{32}{3} \left\{ \tau \left( \frac{7}{4} - \ln \tau \right) + (1 + \tau) \left[ L_2(-\tau) + \ln \tau \ln(1 + \tau) \right] \right\}, \tag{11}$$

$$\widehat{w}_>(\tau) = \frac{32}{3} \left\{ 1 + \ln \tau + \left( \frac{3}{4} + \frac{1}{2} \ln \tau \right) \frac{1}{\tau} + (1 + \tau) \left[ L_2(-\tau^{-1}) - \ln \tau \ln(1 + \tau^{-1}) \right] \right\},$$

where $L_2(x) = - \int_0^x \frac{\mathrm{d}t}{t} \ln(1 - t)$ is the dilogarithm. The physical interpretation of $\widehat{w}$ as the distribution of the internal gluon virtualities in Feynman diagrams was also pointed out in [11].

The function $\widehat{w}$ can be also calculated explicitly for a finite number of renormalons with branch-point singularities. For instance, for the Borel transform written in (9) we obtain [18]:

$$\widehat{w}_<(\tau) = \frac{K}{\Gamma(\nu + 1)}\, p^{\nu+1}\, \tau^{p-1} \left( -\ln \tau \right)^{\nu}, \tag{12}$$

$$\widehat{w}_>(\tau) = \frac{K'}{\Gamma(\nu' + 1)}\, (p')^{\nu'+1}\, \frac{1}{\tau^{p'+1}} \left( \ln \tau \right)^{\nu'}.$$

We now proceed to the evaluation of the integrals (3), taking $a$ in the right half plane, $\mathrm{Re}\, a > 0$, where we assume that they converge. Our aim is to obtain a representation of

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$\mathcal{D}_{PT}^{(\pm)}(a)$ in terms of the inverse Mellin transform $\widehat{w}$. To this end we rotate the integration contours $\mathcal{C}_{\pm}$, whitout crossing singularities, up to a line parallel to the imaginary axis, where the representation (6) is valid. It is easy to check that for $a$ in the upper half plane ($\text{Im}\, a > 0$), the contribution of the quarter of the circle at infinity vanishes if the rotation is performed in the upper half of the $u$-plane. Using the representation (6), valid along the imaginary axis, and performing the integral with respect to $u$ (for details see [6]), we obtain

$$\mathcal{D}_{PT}^{(+)}(a) = \frac{1}{\beta_0} \int\limits_0^\infty \frac{\widehat{w}(\tau)}{\frac{1}{\beta_0 a} + \ln \tau} d\tau, \quad \text{Im}\, a > 0\,. \tag{13}$$

Similarly, for $a$ in the lower half plane ($\text{Im}\, a < 0$) the integration axis in the expression of $\mathcal{D}_{PT}^{(-)}(a)$ can be rotated in the lower half of the $u$-plane, up to the negative imaginary axis, leading to

$$\mathcal{D}_{PT}^{(-)}(a) = \frac{1}{\beta_0} \int\limits_0^\infty \frac{\widehat{w}(\tau)}{\frac{1}{\beta_0 a} + \ln \tau} d\tau, \quad \text{Im}\, a < 0\,, \tag{14}$$

We notice further, recalling the definition (3), that the first relation (10) can be expressed as

$$\mathcal{D}_{PT}^{(+)}(a) = \mathcal{D}_{PT}^{(-)}(a) + 2i\,\sigma_<(a) \tag{15}$$

where we introduced the notation

$$\sigma(a) = \frac{\pi}{\beta_0} [\tau\widehat{w}(\tau)]_{\tau=e^{-\frac{1}{\beta_0 a}}}\,. \tag{16}$$

In particular, using the expression (12) of $\widehat{w}$, we obtain for the leading renormalons:

$$\sigma_<(a) = K\, \frac{2^{\nu+1}}{\Gamma(\nu+1)} \frac{\pi}{\beta_0} e^{-\frac{2}{\beta_0 a}} (\beta_0 a)^{-\nu}\,, \tag{17}$$
$$\sigma_>(a) = K'\, \frac{1}{\Gamma(\nu'+1)} \frac{\pi}{\beta_0} e^{\frac{1}{\beta_0 a}} (-\beta_0 a)^{-\nu'}\,.$$

It is important to emphasize that the relation (15) is valid for $\text{Re}\, a > 0$ (or equivalently for $|\tau| < 1$), i.e. in the whole right-half of the $a$-plane.

The relations (13), (14) and (15) are the basis the derivation of the dispersion relations given below. We first note that, unlike the original representations (3) which converge only for $\text{Re}\, a > 0$, the representations (13) and (14) can be analytically continued in the corresponding upper (lower) half of the complex $a$ plane, outside the real axis. Moreover, it is easy to convert them into a dispersion representation in the variable $a$. We first split

the integral in two integrals, one from 0 to 1 (where $\widehat{w} = \widehat{w}_<$), and the other from 1 to $\infty$ (where $\widehat{w} = \widehat{w}_>$), and perform in each interval the change of variable

$$\ln \tau = -\frac{1}{\beta_0 a'}, \quad \frac{\mathrm{d}\tau}{\tau} = \frac{\mathrm{d}a'}{\beta_0 (a')^2}. \tag{18}$$

Using the relations (13)-(14) thus transformed, together with (15) and (16), we finally express $\mathcal{D}_{PT}^{(+)}(a)$ as

$$\mathcal{D}_{PT}^{(+)}(a) = \frac{a}{\pi} \int_0^\infty \frac{\sigma_<(a')\,\mathrm{d}a'}{a'(a'-a)} + \frac{a}{\pi} \int_{-\infty}^0 \frac{\sigma_>(a')\,\mathrm{d}a'}{a'(a'-a)}, \quad \mathrm{Im}\, a > 0, \tag{19}$$

$$\mathcal{D}_{PT}^{(+)}(a) = \frac{a}{\pi} \int_0^\infty \frac{\sigma_<(a')\,\mathrm{d}a'}{a'(a'-a)} + \frac{a}{\pi} \int_{-\infty}^0 \frac{\sigma_>(a')\,\mathrm{d}a'}{a'(a'-a)} + 2i\sigma_<(a), \quad \mathrm{Im}\, a < 0.$$

From the definition (16) and the properties of the inverse Mellin transform $\widehat{w}$, it follows that the function $\sigma_<(a)$ can be analytically continued in the complex $a$-plane. This property is seen explicitly in the case of one infrared renormalon in (17). Thus, the expressions (19) are analytic functions in the upper (lower) half of the complex $a$-plane, outside the real axis. From (17) it is seen that $\sigma_>(a)$ is real for $a < 0$, while $\sigma_<(a)$ is real for $a > 0$. Therefore, the spectral functions of the dispersion integrals are real.

Taken together, the dispersion relations (19) define a single analytic function, $\mathcal{D}_{PT}^{(+)}(a)$, in the whole cut $a$-plane. This function may have a discontinuity across the positive axis, due the Cauchy dispersion integral along $a > 0$ and the additional term $2i\sigma_<(a)$ in the second relation. A simple calculation shows however that

$$\lim_{\epsilon \to 0_+} \mathcal{D}_{PT}^{(+)}(a \pm i\epsilon) = \frac{a}{\pi} \mathrm{P} \int_0^\infty \frac{\sigma_<(a')\,\mathrm{d}a'}{a'(a'-a)} + \frac{a}{\pi} \int_{-\infty}^0 \frac{\sigma_>(a')\,\mathrm{d}a'}{a'(a'-a)} + i\sigma_<(a), \quad a > 0, \tag{20}$$

where P denotes the Cauchy principal value. This relation shows that the function $\mathcal{D}_{PT}^{(+)}(a)$ is well defined along the positive axis, but has there an unphysical imaginary part equal to

$$\mathrm{Im}\, \mathcal{D}_{PT}^{(+)}(a) = \sigma_<(a), \quad a > 0. \tag{21}$$

From (19) it follows that $\mathcal{D}_{PT}^{(+)}(a)$ has actually a discontinuity along the negative axis, given by

$$\mathcal{D}_{PT}^{(+)}(a + i\epsilon) - \mathcal{D}_{PT}^{(+)}(a - i\epsilon) = 2i\left[\sigma_>(a) - \sigma_<(a - i\epsilon)\right], \quad a < 0, \tag{22}$$

in terms of the real spectral function $\sigma_>$ and the analytic continuation of $\sigma_<$ up to the lower edge of the negative semiaxis (where, as seen from (17), it is complex).

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The function $\mathcal{D}_{PT}^{(-)}(a)$ satisfies a dispersion relation similar to (19), with an additional term $-2i\sigma_<(a)$ in the right hand side, as follows from (15). As above, one can show that $\mathcal{D}_{PT}^{(-)}(a)$ is well-defined for $a > 0$, but it assumes there complex values. The unphysical imaginary part is eliminated if we adopt the principal value prescription defined in (4), for which we obtain:

$$\mathcal{D}_{PT}^{(PV)}(a) = \frac{a}{\pi} \int_0^\infty \frac{\sigma_<(a')\,\mathrm{d}a'}{a'(a'-a)} + \frac{a}{\pi} \int_{-\infty}^0 \frac{\sigma_>(a')\,\mathrm{d}a'}{a'(a'-a)} - i\sigma_<(a), \quad \mathrm{Im}\,a > 0 \qquad (23)$$

$$\mathcal{D}_{PT}^{(PV)}(a) = \frac{a}{\pi} \int_0^\infty \frac{\sigma_<(a')\,\mathrm{d}a'}{a'(a'-a)} + \frac{a}{\pi} \int_{-\infty}^0 \frac{\sigma_>(a')\,\mathrm{d}a'}{a'(a'-a)} + i\sigma_<(a), \quad \mathrm{Im}\,a < 0.$$

For $a$ on the positive semiaxis, is easy to check that

$$\lim_{\epsilon \to 0_+} \mathcal{D}_{PT}^{(PV)}(a + i\epsilon) = \frac{a}{\pi} P \int_0^\infty \frac{\sigma_<(a')\mathrm{d}a'}{a'(a'-a)} + \frac{a}{\pi} \int_{-\infty}^0 \frac{\sigma_>(a')\,\mathrm{d}a'}{a'(a'-a)}, \quad a > 0. \qquad (24)$$

Therefore, $\mathcal{D}_{PT}^{(PV)}(a)$ is well-defined and real on the positive real semiaxis, as required by general principles. The expressions (23) define an analytic function which satisfies the reality condition $\mathcal{D}_{PT}^{(PV)}(a^*) = [\mathcal{D}_{PT}^{(PV)}(a)]^*$ in the whole complex $a$-plane cut along the negative semiaxis, where it has a discontinuity

$$\mathcal{D}_{PT}^{(PV)}(a + i\epsilon) - \mathcal{D}_{PT}^{(PV)}(a - i\epsilon) = 2i[\sigma_>(a) - \mathrm{Re}\,\sigma_>(a)], \quad a < 0. \qquad (25)$$

As argued in [6], this discontinuity vanishes only under strong restrictions on the asymptotic behavior of $B(u)$, which are satisfied neither in the simple cases considered here, nor, most probably, in full QCD: namely, the $L^2$ condition (7) must be satisfied not only by $B(u)$, but also by the product $B(u) \sin \pi u$ (for technical details, see Ref. [6]).

We point out that the imaginary part of $\mathcal{D}_{PT}^{(PV)}(a)$ along the positive semiaxis vanished due to a precise cancellation of the imaginary part of the integrals and the last terms in Eqs. (23). It is easy to check that for a general linear combination of $\mathcal{D}_{PT}^{(\pm)}$, with coefficients $\xi$ and $1 - \xi$ as discussed above (4), this cancellation no longer holds. Therefore, the principal value prescription is the most suitable choice if one wants to preserve in perturbative QCD the analytic properties of the true amplitudes.

In this section, we obtained the nontrivial result that the inverse Mellin transform can be used to derive from (3) dispersion relations for the perturbative amplitudes $\mathcal{D}_{PT}^{(\pm)}(a)$, which allow an analytic continuation of the Borel integral into the left-hand half-plane $\mathrm{Re}\,a < 0$ and explicitly exhibit the singularities and the discontinuities across the cuts.

The results obtained here, besides expressing the Borel integral (3) in the more suitable form of a dispersion relation, will be useful in discussing the validity of some assumptions on analyticity in the $a$ plane made in the literature, as we show in the next section.

## III. COMMENTS

In this section we shall make a few comments on the recent papers [3, 4, 5], related to the present work.

In Ref. [4] the author considers, in the case of a single infrared renormalon, the problem of removing the unphysical imaginary part of $\mathcal{D}_{PT}^{(+)}(a)$ by subtracting a suitable regularization function from it. Our results in this particular case are consistent with [4]. Indeed, Eqs. (19) and (23) show that the function $\mathcal{D}_{PT}^{(PV)}(a)$ is obtained by subtracting from $\mathcal{D}_{PT}^{(+)}(a)$ the function

$$\Delta(a) = i\sigma_<(a)\,, \tag{26}$$

which, using (17), can be written for $a = |a|e^{i\psi}$ as

$$\Delta(a) = K\,\frac{2^{\nu+1}}{\Gamma(\nu+1)}\left(\frac{\pi}{\beta_0}\right)e^{-\frac{2}{\beta_0 a}}\,|\beta_0 a|^{-\nu}[\sin\psi\nu + i\cos\psi\nu]\,. \tag{27}$$

This expression coincides with Eq. (17) of [4], derived using arguments based on regularity with respect to the parameter $\nu$.

The analytic structure in the coupling constant plane was recently considered also in Ref. [3], where arguments based on analyticity were used in support of the claim that the QCD condensates can be determined using the coefficients of the perturbation series. In what follows we shall briefly analyse the validity of this claim.

The author of [3] uses the analogy with some semiclassical models [19], where a specific contribution (for instance, multi-instantons) is regular for negative couplings and can be obtained for positive couplings by analytic continuation. He invokes the heuristic argument according to which a perturbation series with a sign-nonalternating, $n!$ large order behavior, can be summed at $a < 0$ by a Borel integral along the negative axis in the $u$-plane, where it becomes sign alternating. We note however that sign alternation does not necessarily imply Borel summability: this property is violated if a nonalternating component is present, however negligible it may be in comparison with a strong sign-alternating component of the

series. This is exactly the situation in QCD: due to the ultraviolet renormalons, the Borel integral along the negative $u$-axis is not well defined.

Another conjecture adopted in [3] is related to the nonperturbative amplitude $\mathcal{D}_{NP}(a)$, which must be added to $\mathcal{D}_{PT}(a)$ in order to compensate its unphysical imaginary part:

$$\operatorname{Im}\mathcal{D}_{NP}(a+i\epsilon) + \operatorname{Im}\mathcal{D}_{PT}(a+i\epsilon) = 0, \quad a > 0. \tag{28}$$

According to current interpretations [9], the cancellation is expected to occur if both terms in the above relation are calculated with the same prescription. In [3] the author supplements (28) by a specific assumption about the nonperturbative amplitude, taking it of the form

$$\mathcal{D}_{NP}(a) = C\, e^{-\frac{2}{\beta_0 a}} (-\beta_0\, a)^{-\nu}, \tag{29}$$

where $\nu$ is the branch-point exponent of the infrared renormalon in (9) and $C$ is related to the gluon condensate (we use the parametrization given in [5], with our notation $a = \alpha_s/\pi$). This expression is regular for negative $a$, having a discontinuity *only* on the positive semiaxis in the $a$-plane, where

$$\mathcal{D}_{NP}(a \pm i\epsilon) = C\, e^{-\frac{2}{\beta_0 a}} (\beta_0\, a)^{-\nu} \left[\cos \pi\nu \mp i \sin \pi\nu\right], \quad a > 0. \tag{30}$$

Using this relation and the imaginary part of the perturbative amplitude given in (21) and (17), condition (28) gives the relation

$$K = C\, \frac{1}{2^{\nu+1}} \frac{\beta_0}{\Gamma(-\nu)}, \tag{31}$$

from which, according to [3], one could obtain the nonperturbative parameter $C$ (the gluon condensate), using the strength $K$ of the infrared renormalon computed from the perturbation series.

The relation (31) implies that $K$ vanishes for nonnegative integer $\nu$, when the renormalon $1/(1-u/2)^{1+\nu}$ becomes a pole. It is known however that poles are actually obtained from some chains of Feynman diagrams, in the large-$\beta_0$ (or large-$n_f$) limit [14, 15]. In [5] the author discusses this limit, taking the exponent $\nu$ of the form $\nu = \kappa + \chi/\beta_0 + \ldots$, with $\kappa$ an integer. In this limit, the factor $\Gamma(-\nu) \sim \beta_0/\chi$ in the denominator of (31) is compensated by the factor $\beta_0$ in the numerator, leading to a finite nonzero limit for $K$, if $C$ tends to a nonzero constant in the large-$\beta_0$ limit. But then Eq. (30) implies that the real part of the nonperturbative amplitude $\mathcal{D}_{NP}$ is nonvanishing in this limit (we keep, following [5], the

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product $\beta_0 a$ constant, which is legitimate, as seen in particular in the one-loop expression $a = \frac{1}{\beta_0 \ln Q^2/\Lambda^2}$). On the other hand, the perturbative amplitude $\mathcal{D}_{PT}$ is subleading for large $\beta_0$, due to the factor $1/\beta_0$ in front of the Laplace-Borel integral (3). A choice of the constant $C$ of the same order, i.e. $C \sim c/\beta_0$, would imply from (31) that $K$ vanishes when $\beta_0 \to \infty$. So, it follows from [3, 5] that in the large-$\beta_0$ limit either the total Adler function is dominated by the real part of the nonperturbative amplitude, or the renormalon residue vanishes.

The above implications depend however, in a crucial way, on the conjecture made in [3] that the only singularities of the nonperturbative amplitude are along the positive semiaxis. Actually, singularities along the negative semiaxis (produced by the ultraviolet renormalons in the OPE coefficients) cannot be excluded. In [3] it is claimed that the influence of the ultraviolet renormalons can be suppressed by an appropriate conformal mapping [16]. A numerical suppression, however, does not imply removal of the corresponding singularities. As is shown in [8], even if $B(u)$ is expanded in powers of the optimal conformal mapping of the Borel plane [20], the function $\mathcal{D}_{PT}(a)$ does have singularities along the negative semiaxis. Therefore, the general expression of $\mathcal{D}_{NP}(a)$ may have singularities both for $a > 0$ and for $a < 0$. To give an example, we add to (30) the analogous branch-point term suggested by Eq.(26), taking

$$\mathcal{D}_{NP}(a) = C\, e^{-\frac{2}{\beta_0 a}}(-\beta_0\, a)^{-\nu} + C'\, e^{-\frac{2}{\beta_0 a}}(\beta_0\, a)^{-\nu}\,, \tag{32}$$

where $C$ and $C'$ depend in general on $\nu$ and $C' = C'_R + iC'_I$ is complex (as the condition (28) holds in a definite prescription, we specifically refer here to the prescription leading to $\mathcal{D}_{PT}^{(+)}$). Then condition (28) gives

$$-C\,\sin\pi\nu + C'_I + K\,\frac{2^{\nu+1}}{\Gamma(\nu+1)}\,\frac{\pi}{\beta_0} = 0\,, \tag{33}$$

which does not necessarily imply that $K$ vanishes when $\nu$ is a nonnegative integer. Moreover, the real part of the nonperturbative amplitude (32) contains the additional parameter $C'_R$ not specified by the condition (33), which, combined with the first term in the expression (32) can make $\mathrm{Re}\,\mathcal{D}_{NP}$ subleading in the large-$\beta_0$ limit. This counterexample disproves the conclusions (that appear to be inherent in the method of [3, 5]) that the renormalon residue vanishes when the singularity is a pole, or the amplitude is dominated in the large-$\beta_0$ limit by the nonperturbative part. Even more important, however, is the fact that the real part of the nonperturbative amplitude cannot be determined from the requirement (28) and analyticity arguments. The possibility, advocated in [3], of obtaining the gluon condensate

from the perturbation series fails to work here.

In the present paper, by using the inverse Mellin transform, we showed that the Borel-summed QCD perturbative amplitudes (3) satisfy dispersion relations which explicitly exhibit the singularities in the complex $a$-plane and the discontinuities across the cuts, and also allow the analytic continuation of the Borel integral into the left-hand half-plane $\mathrm{Re}\, a < 0$. We saw that the derivation relies on some special hypotheses about the properties of the Borel transform in perturbative QCD: we namely assumed that the Borel transform has no singularities in the complex $u$-plane except for branch-points on the real axis, with a holomorphy gap around the origin, its asymptotic behavior being such that the inequality (7) holds and the inverse Mellin transform (5) exists. We point out that the holomorphy of $B(u)$ in the double-cut $u$-plane is expected on general grounds to hold in renormalisable field theories [21, 22]. The validity of this condition in QCD is presently almost universally adopted as a plausible assumption. As for the asymptotic condition, its validity in full QCD is not guaranteed, but it is satisfied in the large-$\beta_0$ limit and for a finite number of renormalons with branch-point singularities.

In the second part of the paper we made some comments on the papers [3, 4, 5], where arguments based on analyticity were used to discuss the connection between renormalons and nonperturbative quantities in QCD. We investigated some specific conjectures made in [3, 5] and showed that they are in conflict with the analyticity properties related to the ultraviolet and infrared renormalons, which throws doubts on the method of calculating QCD condensates from perturbative expansions.

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